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A q-analogue of Bargmann space and its scalar product

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Abstract. A q-analogue of Bargmann space is defined, using the properties of coherent states associated with a pair of q-deformed bosons. The space consists of a class of entire functions of a complex variable z, and has a reproducing kernel. On this space, the q-boson creation and annihilation operators are represented as multiplication by z and q-differentiation with respect to z, respectively. A q-integral analogue of Bargmann's scalar product is defined, involving the q-exponential as a weight function. Associated with this is a completeness relation for the q-coherent states.

1. Introduction

The representation theory of certain one-parameter (q-) deformations of the universal enveloping algebras of simple Lie algebras is currently of great interest in the study of conformal field theories [1], the classification of links [2], integrable lattice systems in statistical mechanics [2, 3] and quantum inverse scattering [4]. These 'quantized' universal enveloping (QUE) algebras, sometimes loosely called 'quantum groups', have many remarkable properties [5], and it seems likely that they will have a role to play in the development of other areas of physics. Indeed, it has already been indicated [6] that $U_q[\mathrm{su}(1,1)]$, the QUE-algebra associated with the simple Lie algebra su(1, 1), may underly a generalization of string theory; and the question arises as to whether there may exist a q-deformation of quantum field theory. While not pursuing that particular question, several authors [7–10] have already described a q-deformed version of the Bose commutation relations.

Thus q-boson creation and annihilation operators $\bar{b}_i$, b_i, $i = 1, 2, \ldots, r$, have been introduced, together with corresponding number operators N_i, satisfying the usual boson relations

$$[b_i, N_j] = \delta_{ij} b_j \qquad [\bar{b}_i, N_j] = -\delta_{ij}\bar{b}_j \qquad [b_i, b_j] = 0 = [\bar{b}_i, \bar{b}_j] \tag{1}$$

but also

$$\begin{aligned} &[b_i, \bar{b}_j] = 0 \qquad && i \neq j \\ &\bar{b}_i b_i = [N_i] \qquad && b_i \bar{b}_i = [N_i + I] \end{aligned} \tag{2}$$

in place of the corresponding boson relations. In (2), I is the identity operator on the Hilbert space where the creation and annihilation operators act and, for example,

$$[N] = \frac{q^N - q^{-N}}{q - q^{-1}} \tag{3}$$

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where q is a complex parameter. Because $[N] \to N$ and $[N+I] \to N+I$ as $q \to 1$, the usual boson relations are recovered in this limit. Note that (2) implies

$$b_i\bar{b}_i - q^{-1}\bar{b}_i b_i = q^{N_i} \qquad b_i\bar{b}_i - q\bar{b}_i b_i = q^{-N_i}. \tag{4}$$

In what follows we shall concentrate on $0 < q < 1$; the range $1 < q < \infty$ then corresponds to the replacement $q \leftrightarrow q^{-1}$ throughout. It follows in particular that we are not concerned with the so-called singular values of q, where $q^K = 1$ for some non-zero integer K.

Just as the boson calculus can be utilized in the representation theory of simple Lie algebras such as su(m) [11], so q-bosons can be used to construct representations of $U_q[(\mathrm{su}(m)]$ [8, 9]. Furthermore, q-coherent states can be constructed [8] as eigenvectors of the annihilation operators b_i. In the usual boson case, it is known that the coherent states provide the means to go from an abstract formalism to Bargmann's realization [12], where the Hilbert space is a reproducing-kernel Hilbert space of entire functions, the creation operators are simply multiplication operators in complex variables z_i, and the annihilation operators are differential operators $\partial/\partial z_i$. Basis vectors for representations of su(m) in such a realization are simply polynomials in the complex variables z_i.

It should be possible to proceed similarly in the q-boson case, and construct an analogue of Bargmann's realization; indeed, such an analogue has partially been described already [10, 13]. There the creation and annihilation operators act as multiplication operators and q-differential operators, respectively, and basis vectors for $U_q[\mathrm{su}(m)]$ appear as polynomials. Such q-differential operators, together with corresponding q-integration rules, have been discussed earlier [6, 14].

However, there are some questions that have not been addressed. In particular, no analogue has yet been found of Bargmann's scalar product, which is associated with a completeness relation for the usual coherent states, and which involves the weight function $\exp(-\Sigma_{i=1}^r |z_i|^2)$. That is the main object of the present work. We shall see that the new scalar product and completeness relation involve the q-exponential [8, 10, 14] in place of the usual one, and q-integration in (partial) place of Riemann integration. Because the q-exponential has very different behaviour from the usual exponential function, the definition of the scalar product and completeness relation are by no means obvious.

2. Bosons, coherent states and Bargmann space

We consider the case that there is just one pair of Hermitian conjugate creation and annihilation operators; the extension to $r > 1$ pairs is straightforward. For ordinary bosons a, $\bar{a}$ satisfying $[a, \bar{a}] = I$, the corresponding number operator is defined by $N = \bar{a}a$, and has normalized eigenvectors $|n\rangle$ for eigenvalues $n = 0, 1, 2 \ldots$.

Coherent states $|z\rangle$ are defined as eigenvectors of the annihilation operator a:

$$a|z\rangle = z|z\rangle \qquad |z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle. \tag{5}$$

These normalizable vectors are defined for all $z \in \mathbb{C}$, and satisfy $\langle z|w\rangle = \exp(z^* w)$. They are overcomplete, and satisfy in particular the completeness relation

$$\frac{1}{\pi}\iint |z\rangle\langle z|\, \mathrm{e}^{-|z|^2}\, \mathrm{d}^2 z = I \tag{6}$$

where the integral is taken over the entire complex plane, with $\mathrm{d}^2 z = \mathrm{d}x\,\mathrm{d}y$.

The mapping to the Bargmann realization is now obtained by identifying each vector $|\phi\rangle$ in Hilbert space with an entire function $\phi(z)$ defined by

$$\phi(z)=\langle z^*|\phi\rangle=\sum_{n=0}^{\infty}\frac{\langle n|\phi\rangle}{\sqrt{n!}}z^n. \tag{7}$$

In this realization, the scalar product $\langle\phi|\psi\rangle$ takes the form

$$\langle\phi|\psi\rangle=(\phi,\psi)=\frac{1}{\pi}\iint\phi(z)^*\psi(z)\,\mathrm{e}^{-|z|^2}\,\mathrm{d}^2z \tag{8}$$

as can be seen with the help of (6). If an entire function ϕ represents a vector $|\phi\rangle$ as in (7) then

$$|\phi(z)|\leqslant A\,\mathrm{e}^{1/2|z|^2}\qquad\frac{1}{\pi}\iint|\phi(z)|^2\,\mathrm{e}^{-|z|^2}\,\mathrm{d}^2z<\infty \tag{9}$$

where $A\geqslant 0$ is a constant, which can in fact be taken equal to $\langle\phi|\phi\rangle^{1/2}$. Conversely, if ϕ is an entire function satisfying (9), then there exists a vector $|\phi\rangle$ in Hilbert space to which ϕ corresponds. Thus, if the power series expansion of ϕ is

$$\phi(z)=\sum_{n=0}^{\infty}c_nz^n \tag{10}$$

then

$$|\phi\rangle=\sum_{n=0}^{\infty}\sqrt{n!}\,c_n|n\rangle. \tag{11}$$

Note in this connection that conditions (9) imply, and are in fact equivalent to, the condition

$$\sum_{n=0}^{\infty}n!|c_n|^2<\infty. \tag{12}$$

In this way, the Hilbert space is represented by the Bargmann space of entire functions ϕ satisfying (9), with scalar product (8). The creation operator is represented by the multiplicative operator z in this realization, and the annihilation operator by differentiation with respect to z. The number state $|n\rangle$ is represented by the monomial $\phi_n(z)=z^n/\sqrt{n!}$, and the coherent state $|w\rangle$ by the function $\phi_w(z)=\exp(wz)$. If ϕ and ψ are two functions in the Bargmann space, with power series expansions as in (10), and expansion coefficients c_n, d_n respectively, then (8) implies that

$$(\phi,\psi)=\sum_{n=0}^{\infty}n!\,c_n^*\,\mathrm{d}n. \tag{13}$$

In particular it is true that $(\phi_m,\phi_n)=\delta_{mn}$; this is most easily checked directly by changing the variable of integration in (8) to polar form $z=r\exp(i\theta)$, so that

$$(\phi_m,\phi_n)=\frac{1}{\pi\sqrt{m!n!}}\int_0^{\infty}\left[\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta\right]r^{m+n+1}\,\mathrm{e}^{-r^2}\,\mathrm{d}r. \tag{14}$$

Note that since a is Hermitian conjugate to $\bar{a}$, then $\mathrm{d}/\mathrm{d}z$ $(=\exp(-\mathrm{i}\theta)\partial/\partial r)$ must be Hermitian cojugate to z $(=\exp(\mathrm{i}\theta)r)$ with respect to the scalar product (8). From (14) we see that this is guaranteed by the result

$$\int_0^{\infty}x^k\,\mathrm{e}^{-x}\,\mathrm{d}x=k!\,. \tag{15}$$

This observation will be relevant when we come to construct a q-analogue of (8).

The Bargmann space has $K(w|z)=\exp(w^*z)$ as a reproducing kernel. Thus

$$(K(w|\cdot),\phi(\cdot))=\frac{1}{\pi}\int\int e^{z^*w}\phi(z)\,e^{-|z|^2}\,d^2z=\phi(w) \tag{16}$$

which can be deduced by expanding ϕ as in (10) and integrating term by term.

3. The q-analogues of bosons, coherent states and Bargmann space

The creation and annihilation operators $\bar{a}$, a can be deformed to a Hermitian conjugate pair of operators $\bar{b}$, b satisfying, as a particular case of (1) and (2),

$$\bar{b}b=[N]\qquad b\bar{b}=[N+I]\qquad [b,N]=b\qquad [\bar{b},N]=-\bar{b}. \tag{17}$$

It is possible to assume that $\bar{b}$, b act in the same space as $\bar{a}$, a, and indeed that $N(=\bar{a}a)$ is the same operator as before, with the same eigenvectors $|n\rangle$. Then it follows from (17), to within unimportant choices of phases, that

$$b|n\rangle=\sqrt{[n]}|n-1\rangle\qquad \bar{b}|n\rangle=\sqrt{[n+1]}|n+1\rangle. \tag{18}$$

Comparing these with the well known actions of a and $\bar{a}$ on $|n\rangle$, we see that

$$b=\frac{\sqrt{[N+I]}}{\sqrt{N+I}}\,a\qquad \bar{b}=\bar{a}\frac{\sqrt{[N+I]}}{\sqrt{N+I}} \tag{19}$$

which shows explicitly the form of the deformation. Note that as $q\to 1$, then $b\to a$ and $\bar{b}\to\bar{a}$.

The q-coherent states are now defined [8] as eigenvectors of the annihilation operator b:

$$b|z;q\rangle=z|z;q\rangle\qquad |z;q\rangle=\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]!}}|n\rangle \tag{20}$$

$$[n]!=[n][n-1]\dots[2][1]\qquad [0]!=1.$$

These vectors are normalizable for all complex z and satisfy

$$\langle z;q|w;q\rangle=E(z^*w;q) \tag{21}$$

where $E(v;q)=\sum_{n=0}^{\infty}v^n/[n]!$ is the q-exponential [8, 10, 14], which we shall write as $E(v)$, treating q as an implicit parameter. The function E can be seen to be entire since $[n]\geqslant n$ for $q>0$, so that $[n]!\geqslant n!$ and $|E(v)|\leqslant\exp(|v|)$.

We can now map each vector $|\phi\rangle$ in the Hilbert space into an entire function ϕ by defining, in place of (7),

$$\phi(z)=\langle z^*;q|\phi\rangle=\sum_{n=0}^{\infty}\frac{\langle n|\phi\rangle}{\sqrt{[n]!}}z^n. \tag{22}$$

Each such entire function $\phi(z)=\sum_{n=0}^{\infty}c_nz^n$ satisfies

$$\sum_{n=0}^{\infty}[n]!|c_n|^2<\infty. \tag{23}$$

Conversely, any entire function ϕ whose expansion coefficients c_n satisfy (23), corresponds to a vector given by

$$|\phi\rangle=\sum_{n=0}^{\infty}\sqrt{[n]!}\,c_n|n\rangle. \tag{24}$$

The vector space of all such functions can be made into a Hilbert space by defining the scalar product of any two functions ϕ, ψ with expansion coefficients c_n, d_n as

$$(\phi, \psi) = \sum_{n=0}^{\infty} [n]!\, c_n^* d_n \tag{25}$$

so that $(\phi, \psi) = \langle\phi|\psi\rangle$. (Completeness of the space follows as in the usual case.)

In this way, we set up a q-analogue of Bargmann space. We can see already from (25) that there is a reproducing kernel for the new space, given by $K(w|z; q) = E(w^*z)$. Note, however, that to this stage we have no analogue of the integral form (8) of the scalar product; this is associated with the fact that we have not yet given an analogue of the completeness relation (6). The forms of such analogues are strongly related to the realizations of the annihilation and creation operators b, $\bar{b}$, since these operators must be conjugate to each other with respect to the new scalar product.

4. Realization of q-bosons and the q-calculus

According to (22), the function corresponding to the vector $\bar{b}|\phi\rangle$ is

$$\langle z^*; q|\bar{b}|\phi\rangle = z\langle z^*; q|\phi\rangle = z\phi(z). \tag{26}$$

It follows that $\bar{b}$ is represented by the multiplicative operator z. To determine the representative of b, we note first that

$$z\langle z^*; q|b|\phi\rangle = \langle z^*; q|\bar{b}b|\phi\rangle = \langle z^*; q|[N]|\phi\rangle. \tag{27}$$

Since $|n\rangle$ is represented by $z^n/\sqrt{[n]!}$, and $N|n\rangle = n|n\rangle$ for $n = 0, 1, 2 \ldots$, then N is represented by the operator $z\,\mathrm{d}/\mathrm{d}z$. Therefore

$$\langle z^*; q|q^N|\phi\rangle = \phi(qz) \qquad \langle z^*; q|q^{-N}|\phi\rangle = \phi(q^{-1}z) \tag{28}$$

and (27) implies

$$z\langle z^*; q|b|\phi\rangle = \frac{\phi(qz) - \phi(q^{-1}z)}{q - q^{-1}}. \tag{29}$$

Thus b is represented by the q-differential operator $d/d(z; q)$, which acts on functions $\phi(z)$ as

$$\frac{\mathrm{d}\phi(z)}{\mathrm{d}(z; q)} = \frac{\phi(qz) - \phi(q^{-1}z)}{z(q - q^{-1})}. \tag{30}$$

For entire functions ϕ, we note that the q-derivative approaches $\mathrm{d}\phi/\mathrm{d}z$ as $q \to 1$, and also that, if $z = r\exp(\mathrm{i}\theta)$, then $\mathrm{d}/\mathrm{d}(z; q) = \exp(-\mathrm{i}\theta)\partial/\partial(r; q)$.

The q-differential (or q-difference) operator has been discussed previously, together with a corresponding q-integration (or q-summation) operator defined by [6, 13, 14]

$$\int f(z)\,\mathrm{d}(z; q) = (q^{-1} - q)\sum_{n=0}^{\infty} q^{2n+1} z f(q^{2n+1}z) + C \qquad 0 < q < 1 \tag{31}$$

where C is an arbitrary constant†. For entire functions $f(z)$, it is easily seen that this q-integral approaches the Riemann integral as $q \to 1$, and also that the operators of

† The most general function $F(z)$ with q-derivative 0 is actually of the form $F(z) = g(\ln z)$ where g is an elliptic function with periods $2\pi i$ and $2 \ln q$, but unless g is constant, then F has an essential singularity at $z = 0$, so that (31) gives the correct formula if we demand that F has at worst a pole at $z = 0$.

q-differentiation and q-integration are inverse to each other:

$$\frac{\mathrm{d}}{\mathrm{d}(z;q)}\int f(z)\,\mathrm{d}(z;q)=f(z)=\int\frac{\mathrm{d}f(z)}{\mathrm{d}(z;q)}\,\mathrm{d}(z;q). \tag{32}$$

Some of the useful properties of q-differentiation, analogous to those of ordinary differentiation, are:

Sum rule:

$$\frac{\mathrm{d}}{\mathrm{d}(z;q)}[f(z)+g(z)]=\frac{\mathrm{d}f(z)}{\mathrm{d}(z;q)}+\frac{\mathrm{d}g(z)}{\mathrm{d}(z;q)}$$

Product rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}(z;q)}[f(z)g(z)]&=\frac{\mathrm{d}f(z)}{\mathrm{d}(z;q)}g(q^{-1}z)+f(qz)\frac{\mathrm{d}g(z)}{\mathrm{d}(z;q)}\\&=\frac{\mathrm{d}f(z)}{\mathrm{d}(z;q)}g(qz)+f(q^{-1}z)\frac{\mathrm{d}g(z)}{\mathrm{d}(z;q)}\end{aligned} \tag{33}$$

Chain rule (special cases):

$$\frac{\mathrm{d}f(z)}{\mathrm{d}(\alpha z;q)}=\frac{1}{\alpha}\frac{\mathrm{d}f(z)}{\mathrm{d}(z;q)}\qquad\alpha\text{ constant}$$

$$\frac{\mathrm{d}f(z^n)}{\mathrm{d}(z;q)}=[n]z^{n-1}\frac{\mathrm{d}f(z^n)}{\mathrm{d}(z^n;q^n)}.$$

Another useful result is

$$\frac{\mathrm{d}f(z)}{\mathrm{d}(z;q^n)}=\frac{1}{[n]}\sum_{k=0}^{n-1}\frac{\mathrm{d}}{\mathrm{d}(z;q)}f(q^{2k-(n-1)}z). \tag{34}$$

Corresponding results for q-integration are:

$$\begin{aligned}&\int f(z)\,\mathrm{d}(\alpha z;q)=\alpha\int f(z)\,\mathrm{d}(z;q)\\&\int f(z^n)\,\mathrm{d}(z^n;q^n)=[n]\int z^{n-1}f(z^n)\,\mathrm{d}(z;q)\\&\int f(z)\,\mathrm{d}(z;q)=\frac{1}{[n]}\sum_{k=0}^{n-1}q^{2k-(n-1)}\int f(q^{2k-(n-1)}z)\,\mathrm{d}(z;q^n).\end{aligned} \tag{35}$$

In particular we see that $\mathrm{d}z^n/\mathrm{d}(z;q)=[n]z^{n-1}$, so that

$$\frac{\mathrm{d}(z^n/\sqrt{[n]!})}{\mathrm{d}(z;q)}=\sqrt{[n]}(z^{n-1}/\sqrt{[n-1]!})\qquad\frac{\mathrm{d}E(wz)}{\mathrm{d}(z;q)}=wE(wz) \tag{36}$$

consistent with the action of b on $|n\rangle$ and $|w;q\rangle$.

5. The q-analogue of the scalar product

By analogy with the usual case, one might guess that the analogue of (8) is

$$(\phi,\psi)=\frac{1}{\pi}\int_0^\infty\left[\int_0^{2\pi}\phi(r\,\mathrm{e}^{i\theta})^*\psi(r\,\mathrm{e}^{i\theta})\,\mathrm{d}\theta\right]rE(-r^2)\,\mathrm{d}(r;q). \tag{37}$$

However, this is inappropriate because the improper q-integral does not converge. The problem lies in the behaviour of $E(x)$ as $x\to-\infty$: whereas $\exp(x)$ decays to 0 in that limit, the sign of $E(x)$ alternates, and $|E(x)|$ actually grows quickly, so that $E(x)$ can be made arbitrarily large positive or negative by choosing a suitable large negative argument. (See figures 1 and 2.) The resolution of the difficulty will ultimately derive from the fact that $E(x)$ does alternate in sign as $x\to-\infty$, and it involves choosing a special sequence of upper limits $r_n=\sqrt{q^n/(1-q^2)}$, $n=-1,-2,\ldots$, of the q-integral in (37), such that $r_n\to\infty$ as $n\to-\infty$; this resultant sequence of q-integrals converges; and positive definiteness of the scalar product is guaranteed. It can be shown that this sequence of upper limits, which is intimately related to the structure of the q-exponential, is uniquely determined.

As demonstrated in the appendix, if $0<q<1$, $n\in\mathbb{Z}$,

$$E\left(\frac{-q^n}{1-q^2}\right)=\sum_{k=0}^{\infty}\frac{q^{(1/2)(n-2k-1)(n-2k-2)}\prod_{l=k+1}^{\infty}(1-q^{2l})}{\sum_{m=0}^{\infty}q^{(1/2)m(m+1)}}>0 \tag{38}$$

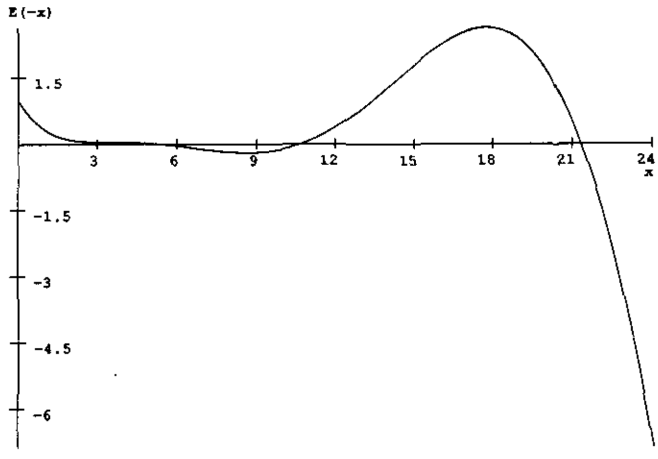

Figure 1. Graph of $E(-x)$ for $q=0.5$.

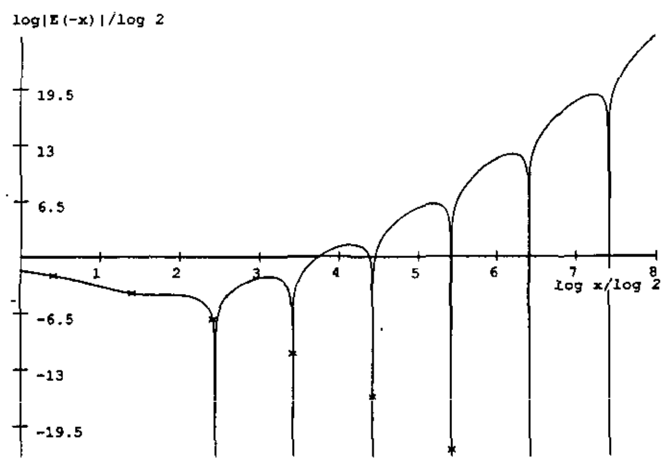

Figure 2. Graph of $\log|E(-x)|/\log 2$ showing the rate of growth. The crosses indicate the special points at which the q-integral is evaluated in (41).

so that

$$E\left(\frac{-q^n}{1-q^2}\right)=\frac{q^{(1/2)(n^2-3n+2)}\prod_{l=1}^{\infty}(1-q^{2l})}{\sum_{m=0}^{\infty}q^{(1/2)m(m+1)}}(1+\mathrm{O}(q^{-2n+5}))$$

$$=\frac{q^{(1/2)(n^2-3n+2)}\sum_{l=-\infty}^{\infty}(-1)^l q^{3l^2-l}}{\sum_{m=0}^{\infty}q^{(1/2)m(m+1)}}(1+\mathrm{O}(q^{-2n+5})) \qquad \text{as } n\to-\infty. \tag{39}$$

Then

$$\left(\frac{q^n}{1-q^2}\right)^k E\left(\frac{-q^n}{1-q^2}\right)$$

$$=\frac{q^{(1/2)(n^2+(2k-3)n+2)}\prod_{l=1}^{\infty}(1-q^{2l})}{(1-q^2)^k\sum_{m=0}^{\infty}q^{(1/2)m(m+1)}}(1+\mathrm{O}(q^{-2n+5}))$$

$$=\frac{q^{(1/2)(n^2+(2k-3)n+2)}\sum_{l=-\infty}^{\infty}(-1)^l q^{3l^2-l}}{(1-q^2)^k\sum_{m=0}^{\infty}q^{(1/2)m(m+1)}}(1+\mathrm{O}(q^{-2n+5})) \qquad \text{as } n\to-\infty.$$

We now argue that the q-analogue of $\int_0^\infty f(x)\exp(-x)\,\mathrm{d}x$ is

$$\lim_{n\to-\infty}\int_0^{q^n/(1-q^2)} f(x)E(-x)\,\mathrm{d}(x;q) \tag{41}$$

where $f(x)$ grows slowly enough that the limit is defined. (In particular, polynomial growth is allowed.) It is important to note from (31) that the only values of $E(-x)$ involved in the evaluation of (41) are those obtained at $x=q^k/(1-q^2)$, for $k=n+1$, $n+2,\ldots$, where $E(-x)$ is positive, as seen from (38).

Using q-integration by parts, we find that

$$\int_0^{q^n/(1-q^2)} x^k E(-x)\,\mathrm{d}(x;q)$$

$$=-\left(\frac{q^{n-1}}{1-q^2}\right)^k E\left(\frac{-q^n}{1-q^2}\right)+[k]\int_0^{(q^{n-1})/(1-q^2)} x^{k-1}E(-x)\,\mathrm{d}(x;q). \tag{42}$$

The first term on the right-hand side goes to zero as $n\to-\infty$, as can be seen from (39). (The rapid decay of $|E(-q^n/(1-q^2))|$ with decreasing $n=0,-1,-2\ldots$ is indicated in figure 2.) Therefore

$$\lim_{n\to-\infty}\int_0^{q^n/(1-q^2)} x^k E(-x)\,\mathrm{d}(x;q)=[k]\lim_{n\to-\infty}\int_0^{q^n/(1-q^2)} x^{k-1}E(-x)\,\mathrm{d}(x;q). \tag{43}$$

A simple proof by induction then shows that

$$\lim_{n\to-\infty}\int_0^{q^n/(1-q^2)} x^k E(-x)\,\mathrm{d}(x;q)=[k]! \tag{44}$$

which is the q-analogue of (15).

We are now in a position to describe a q-analogue of the scalar product (8). Consider again two entire functions ϕ, ψ with power series expansions whose coefficients c_n, d_n satisfy (23); their scalar product is given by (25). First we note that the integral $\int_0^{2\pi}((r\,\mathrm{e}^{i\theta})^k)^*(r\,\mathrm{e}^{i\theta})^l\,\mathrm{d}\theta$ is equal to $2\pi\delta_{kl}r^{2k}$, so that

$$\int_0^{2\pi}\phi(r\,\mathrm{e}^{i\theta})^*\psi(r\,\mathrm{e}^{i\theta})\,\mathrm{d}\theta=\sum_{k=0}^{\infty}c_k^*\,d_k r^{2k}. \tag{45}$$

Also,

$$\int_0^{\sqrt{(q^n/(1-q^2))}} rf(r^2)\,\mathrm{d}(r;q^{1/2}) = \frac{1}{q^{1/2}+q^{-1/2}} \int_0^{q^n/(1-q^2)} f(s)\,\mathrm{d}(s;q) \tag{46}$$

so that

$$\lim_{n\to-\infty} (q^{1/2}+q^{-1/2}) \int_0^{\sqrt{(q^n/(1-q^2))}} r^{2m+1} E(-r^2)\,\mathrm{d}(r;q^{1/2}) = [m]!. \tag{47}$$

Then it follows that

$$\lim_{n\to-\infty} (q^{1/2}+q^{-1/2}) \int_0^{\sqrt{(q^n/(1-q^2))}} \sum_{k=0}^{\infty} c_k^* \, d_k r^{2k+1} E(-r^2)\,\mathrm{d}(r;q^{1/2})$$

$$= \sum_{k=0}^{\infty} c_k^* \, d_k [k]! = (\phi, \psi). \tag{48}$$

The interchange of the q-integration and the sum over k in this step is justified because the double sum is absolutely convergent. A suitable scalar product of two entire functions ϕ and ψ in the q-analogue of Bargmann space is therefore given by

$$(\phi,\psi) = \lim_{n\to-\infty} \left(\frac{q^{1/2}+q^{-1/2}}{2\pi}\right)$$

$$\times \int_0^{\sqrt{(q^n(1-q^2))}} \left[\int_0^{2\pi} \phi(r\,\mathrm{e}^{i\theta})^* \psi(r\,\mathrm{e}^{i\theta})\,\mathrm{d}\theta\right] rE(-r^2)\,\mathrm{d}(r;q^{1/2}). \tag{49}$$

This can also be written in the form

$$(\phi,\psi) = \lim_{n\to-\infty} \left\{ \frac{1}{2\pi} \int_0^{\sqrt{((q^{n+1})/(1-q^2))}} \left[\int_0^{2\pi} \phi(r\,\mathrm{e}^{i\theta})^* \psi(r\,\mathrm{e}^{i\theta})\,\mathrm{d}\theta\right] rE(-r^2)\,\mathrm{d}(r;q)\right.$$

$$\left. + \frac{1}{2\pi} \int_0^{\sqrt{((q^{n-1}/(1-q^2))}} \left[\int_0^{2\pi} \phi(r\,\mathrm{e}^{i\theta})^* \psi(r\,\mathrm{e}^{i\theta})\,\mathrm{d}\theta\right] rE(-r^2)\,\mathrm{d}(r;q)\right\}. \tag{50}$$

In a similar way we can show that the q-analogue of the completeness relation (6) is

$$\lim_{n\to-\infty} \left(\frac{q^{1/2}+q^{-1/2}}{2\pi}\right) \int_0^{\sqrt{(q^n/(1-q^2))}} \left[\int_0^{2\pi} |r\,\mathrm{e}^{i\theta};q\rangle\langle r\,\mathrm{e}^{i\theta};q|\,\mathrm{d}\theta\right] rE(-r^2)\,\mathrm{d}(r;q^{1/2}) = I. \tag{51}$$

This can also be written in a form corresponding to (50).

6. Concluding remarks

The q-analogue of Bargmann space shares many of the properties of Bargmann space itself, which is evidently recovered in the limit $q \to 1$. We may expect that this space, and its generalization to r variables $z_1, z_2, \ldots, z_r$, will prove of value in the representation theory of QUE-algebras, such as $U_q[\mathrm{su}(m,n)]$ in particular, just as the Bargmann space has [15] in the case of the Lie algebras $\mathrm{su}(m,n)$. Indeed, some advantages of the formalism are already clear from earlier studies [10, 13].

We have given a completeness relation for the q-coherent states $|z; q\rangle$. They are evidently overcomplete, and an interesting problem is to find the analogue of von Neumann's result [16], that the ordinary coherent states $|z\rangle$ are complete when restricted to a lattice in the z-plane, provided the lattice spacing is not too large. In this connection, the generalized undertainty relations associated with the q-bosons, as discussed by Biedenharn [8], are certain to play a role.

One of the interesting features of QUE algebra theory is that the introduction of the variable q not only enriches structural properties but also throws new light on (and may even uncover) properties that already hold in the undeformed limit ($q \to 1$). Therefore we may also hope that the extension of the concept of Bargmann space in the way we have described, will lead to new insights into the properties of such spaces.

Finally, we believe that the appearance of the q-derivative and q-integral in the setting of a reproducing-kernel Hilbert space of entire functions may have important implications for the study of q-analogues of special functions, and the subject of q-series analysis [14] in general.

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Appendix

Suppose that $0<q<1$, and let $\{b_n : n \in \mathbb{Z}\}$ be a q-Fibonacci sequence satisfying the linear difference equation

$$b_{n+2} = b_n + q^n b_{n+1}. \tag{A1}$$

Regarded as a real vector space, the set of such sequences is two-dimensional because the difference equation is second-order. Note that the half-sequence $\{b_n : n \geqslant 0\}$ is bounded. To see this, note first that $|b_0|, |b_1| < K_1$ implies

$$|b_n| < 2K_1(1+q)(1+q^2)\dots(1+q^{n-2}) = 2K_1 \prod_{l=1}^{n-2}(1+q^l) \qquad n \geqslant 2. \tag{A2}$$

The proof is by induction on n. Then we have

$$|b_k| < 2K_1 \prod_{l=1}^{k-2}(1+q^l) < 2K_1 \prod_{l=1}^{\infty}(1+q^l) = K_2 \qquad \text{(say)}.$$

Since the product $\Pi_{l=1}^{\infty}(1+q^l)$ converges, the sequence $\{b_n : n \geqslant 0\}$ is bounded.

Now, for k, $l \geqslant 1$,

$$|b_{k+2l} - b_k| = \left| \sum_{m=0}^{l-1} q^{k+2m} b_{k+2m+1} \right| \leqslant \sum_{m=0}^{l-1} q^{k+2m} |b_{k+2m+1}| < \frac{q^k K_2}{1-q^2}. \tag{A3}$$

It follows that the sequences $\{b_{2l} : l \geqslant 0\}$ and $\{b_{2l+1} : l \geqslant 0\}$ are Cauchy, since if k_1, $k_2 \geqslant N$ and are of the same parity, then

$$|b_{k_1} - b_{k_2}| < \frac{q^N K_2}{1-q^2}. \tag{A4}$$

If we let $c_k = E(-q^k/(1-q^2))$, then it follows from property (36) of $E(v)$ that $\{c_k\}$ is a q-Fibonacci sequence. Furthermore, $\lim_{l\to\infty} c_{2l} = \lim_{l\to\infty} c_{2l+1} = E(0) = 1$, so that the sequence $\{E(-q^k/(1-q^2)) \colon n \in \mathbb{Z}\}$ is a q-Fibonacci sequence with equal limits.

Now, suppose that $K \in \mathbb{Z}$ and $K > \ln(1-q^2)/\ln q$, and let $\varepsilon = q^K/(1-q^2)$, so that $0 < \varepsilon < 1$. Let $\{d_n : n \in \mathbb{Z}\}$ be a q-Fibonacci sequence such that $d_K = 1$, $d_{K+1} = -1$. By induction on l, we can prove that

$$1 \geqslant d_{K+2l} > 1 - \varepsilon(1-q^{2l}) \qquad -1 \leqslant d_{K+2l+1} < -1 + \varepsilon q(1-q^{2l}). \tag{A5}$$

This can be shown as follows:

$$\begin{aligned} d_{K+2l+2} &= d_{K+2l} + q^{K+2l} d_{K+2l+1} < d_{k+2l} \leqslant 1 \\ d_{K+2l+2} &= d_{K+2l} + q^{K+2l} d_{K+2l+1} > 1 - \varepsilon(1-q^{2l}) - \varepsilon q^{2l}(1-q^2) \\ &= 1 - \varepsilon(1-q^{2(l+1)}) \\ d_{K+2l+3} &= d_{K+2l+1} + q^{K+2l+1} d_{K+2l+2} > d_{k+2l+1} \geqslant -1 \\ d_{K+2l+3} &= d_{K+2l+1} + q^{K+2l+1} d_{K+2l+2} < -1 + \varepsilon q(1-q^{2l}) - \varepsilon q^{2l+1}(1-q^2) \\ &= -1 + \varepsilon q(1-q^{2(l+1)}). \end{aligned}$$

From (A5) we have

$$1 \geqslant d_{K+2l} > 1 - \varepsilon \qquad -1 \leqslant d_{K+2l+1} < -1 + \varepsilon q \tag{A6}$$

and $\lim_{l\to\infty} d_{2K+l} \geqslant 1 - \varepsilon > 0$, $\lim_{l\to\infty} d_{2K+l+1} \leqslant -1 + \varepsilon q < 0$. Then $\{d_n : n \in \mathbb{Z}\}$ is a q-Fibonacci sequence with distinct limits. Because the space of q-Fibonacci sequences is two-dimensional, it now follows that if $\{b_n : n \in \mathbb{Z}\}$ is a q-Fibonacci sequence with equal limits α, then $b_n = \alpha E(-q^n/(1-q^2))$.

Let

$$a_n = \sum_{k=0}^{\infty} \frac{q^{(n-2k-1)(n-2k-2)/2}}{\prod_{l=1}^{k} (1-q^{2l})}. \tag{A7}$$

Then $\{a_n : n \in \mathbb{Z}\}$ is a q-Fibonacci sequence, $a_n > 0$ for all $n \in \mathbb{Z}$, and $\lim_{l\to\infty} a_{2l} = \lim_{l\to\infty} a_{2l+1} = A$, where

$$A = \frac{\sum_{n=0}^{\infty} q^{n(n+1)/2}}{\prod_{l=1}^{\infty} (1-q^{2l})} = \frac{\sum_{n=0}^{\infty} q^{n(n+1)/2}}{\sum_{m=-\infty}^{\infty} (-1)^m q^{3m^2-m}}. \tag{A8}$$

Therefore

$$a_n = \frac{\sum_{p=0}^{\infty} q^{p(p+1)/2}}{\prod_{l=1}^{\infty} (1-q^{2l})} E\left(\frac{-q^n}{1-q^2}\right) = \frac{\sum_{p=0}^{\infty} q^{p(p+1)/2}}{\sum_{m=-\infty}^{\infty} (-1)^m q^{3m^2-m}} E\left(\frac{-q^n}{1-q^2}\right). \tag{A9}$$

These results for $\{a_n\}$ can be seen as follows: Firstly, the series for a_n converges by the Ratio Test:

$$\frac{\left(\dfrac{q^{(n-2k-1)(n-2k-2)/2}}{\prod_{l=1}^{k} (1-q^{2l})}\right)}{\left(\dfrac{q^{(n-2k+1)(n-2k)/2}}{\prod_{l=1}^{k-1} (1-q^{2l})}\right)} = \frac{q^{-2n+4k+1}}{1-q^{2k}} \to 0 \qquad \text{as } k \to \infty \tag{A10}$$

since $q^{4k}\to 0$, $1-q^{2k}\to 1$. Secondly, $\{a_n : n\in\mathbb{Z}\}$ is a q-Fibonacci sequence since

$$\begin{aligned}
a_{n+2}-a_n &= q^{n(n+1)/2}+\sum_{k=1}^{\infty}\frac{q^{(n-2k+1)(n-2k)/2}}{\Pi_{l=1}^{k}(1-q^{2l})}-\sum_{k=1}^{\infty}\frac{q^{(n-2k+1)(n-2k)/2}}{\Pi_{l=1}^{k-1}(1-q^{2l})}\\
&= q^{n(n+1)/2}+\sum_{k=1}^{\infty}\frac{q^{2k+(n-2k+1)(n-2k)/2}}{\Pi_{l=1}^{k}(1-q^{2l})}\\
&= q^n a_{n+1}.
\end{aligned}\tag{A11}$$

Now,

$$\begin{aligned}
a_{2p} &= \sum_{k=0}^{\infty}\frac{q^{(2p-2k-1)(p-k-1)}}{\Pi_{l=1}^{k}(1-q^{2l})}=\sum_{m=-\infty}^{p-1}\frac{q^{m(2m+1)}}{\Pi_{l=1}^{p-m-1}(1-q^{2l})}\\
a_{2p+1} &= \sum_{k=0}^{\infty}\frac{q^{(p-k)(2p-2k-1)}}{\Pi_{l=1}^{k}(1-q^{2l})}=\sum_{m=-p}^{\infty}\frac{q^{m(2m+1)}}{\Pi_{l=1}^{p+m}(1-q^{2l})}
\end{aligned}\tag{A12}$$

and

$$\begin{aligned}
A &= \frac{\Sigma_{n=0}^{\infty}q^{n(n+1)/2}}{\Pi_{l=1}^{\infty}(1-q^{2l})}=\frac{\Sigma_{m=0}^{\infty}q^{m(2m+1)}+\Sigma_{m=-\infty}^{-1}q^{m(2m+1)}}{\Pi_{l=1}^{\infty}(1-q^{2l})}\\
&= \frac{\Sigma_{m=-\infty}^{\infty}q^{m(2m+1)}}{\Pi_{l=1}^{\infty}(1-q^{2l})}.
\end{aligned}\tag{A13}$$

Therefore

$$0\leqslant A-a_{2p}=\sum_{m=-\infty}^{p-1}\frac{q^{m(2m+1)}}{\Pi_{l=1}^{\infty}(1-q^{2l})}\left(1-\prod_{l=p-m}^{\infty}(1-q^{2l})\right)+\sum_{m=p}^{\infty}\frac{q^{m(2m+1)}}{\Pi_{l=1}^{\infty}(1-q^{2l})}.\tag{A14}$$

Since

$$\prod_{l=K}^{K+L}(1-q^{2l})\geqslant 1-\frac{q^{2K}(1-q^{2(L+1)})}{1-q^2}\tag{A15}$$

(which can be proved by induction on L), then

$$1-\prod_{l=K}^{\infty}(1-q^{2l})\leqslant\frac{q^{2K}}{1-q^2}.\tag{A16}$$

Therefore

$$\begin{aligned}
0\leqslant A-a_{2p} &\leqslant \sum_{m=-\infty}^{p-1}\frac{q^{m(2m+1)}}{\Pi_{l=1}^{\infty}(1-q^{2l})}\frac{q^{2p-2m}}{1-q^2}+\sum_{m=p}^{\infty}\frac{q^{m(2m+1)}}{\Pi_{l=1}^{\infty}(1-q^{2l})}\\
&\leqslant \sum_{m=-\infty}^{p-1}\frac{q^{2m^2-m+2p}}{(1-q^2)\Pi_{l=1}^{\infty}(1-q^{2l})}+\sum_{m=p}^{\infty}\frac{q^{2m^2+m}}{\Pi_{l=1}^{\infty}(1-q^{2l})}\frac{q^{2p-2m}}{1-q^2}\\
&= q^{2p}\frac{\Sigma_{m=-\infty}^{\infty}q^{2m^2-m}}{(1-q^2)\,\Pi_{l=1}^{\infty}(1-q^{2l})}\to 0 \qquad \text{as } p\to\infty
\end{aligned}\tag{A17}$$

where we have used the fact that if $m\geqslant p$, then $q^{2p-2m}/(1-q^2)\geqslant 1$. Similarly,

$$0\leqslant A-a_{2p+1}\leqslant q^{2p+1}\frac{\Sigma_{m=-\infty}^{\infty}q^{2m^2-m}}{(1-q^2)\,\Pi_{l=1}^{\infty}(1-q^{2l})}\to 0 \qquad \text{as } p\to\infty.\tag{A18}$$

Note that $\sum_{m=-\infty}^{\infty} q^{2m^2-m}/((1-q^2)\prod_{l=1}^{\infty}(1-q^{2l}))$ is finite since the product in the denominator converges, and the numerator converges by the Integral Test (using the integrability of Gaussian functions). Thus a_{2p}, $a_{2p+1}\to A$, and (A9) follows. Then we have

$$E\left(\frac{-q^n}{1-q^2}\right)=\sum_{k=0}^{\infty}\frac{q^{(1/2)(n-2k-1)(n-2k-2)}\prod_{l=k+1}^{\infty}(1-q^{2l})}{\sum_{m=0}^{\infty}q^{(1/2)m(m+1)}}. \tag{A20}$$

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